

AN EFFECTIVE COMPACTNESS THEOREM FOR COXETER GROUPS

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ABSTRACT. Through highly non-constructive methods, works by Bestvina, Culler, Feighn, Morgan, Rips, Shalen, and Thurston show that if a finitely presented group does not split over a virtually solvable subgroup, then the space of its discrete and faithful actions on \mathbb{H}^n , modulo conjugation, is compact for all dimensions. Although this implies that the space of hyperbolic structures of such groups has finite diameter, the known methods do not give an explicit bound. We establish such a bound for Coxeter groups. We find that either the group splits over a virtually solvable subgroup or there is a constant C and a point in \mathbb{H}^n that is moved no more than C by any generator. The constant C depends only on the number of generators of the group, and is independent of the relators.

INTRODUCTION

The space of discrete and faithful actions of a given group G on \mathbb{H}^n , up to conjugation, is a *deformation space* of the group. It is denoted $\mathcal{D}(G, n)$. In the 1980's, Thurston proved that when a group G is the fundamental group of an orientable, compact, irreducible, acylindrical 3-manifold with boundary, the deformation space $\mathcal{D}(G, 3)$ is compact [Thu86]. To prove this result, Thurston analysed sequences of ideal triangulations.

Inspired by Thurston's work and Culler-Shalen's work on varieties of three-manifold groups [CS83], Morgan-Shalen reproved Thurston's compactness theorem using methods from algebraic geometry and geometric topology [MS84] [MS88a] [MS88b]. Morgan then showed that when G is the fundamental group for a compact, orientable, and irreducible 3-manifold, the space $\mathcal{D}(G, n)$ is compact if and only if the group G does not admit a virtually abelian splitting [Mor86]. This result was pushed to include all finitely-presented groups using the Rips Machine by Bestvina-Feighn, who state the following Compactness Theorem as a consequence of the main result of [BF95] concerning actions of trees:

Compactness Theorem for Finitely Presented Groups (Thurston, Morgan-Shalen, Morgan, Rips, Bestvina-Feighn). *If G be a finitely-presented group that is not virtually abelian and does not split over a virtually solvable subgroup, then $\mathcal{D}(G, n)$ is compact.*

If a finitely-presented group does not split over a virtually solvable subgroup, then the Compactness Theorem implies that there is a point in \mathbb{H}^n that is not moved too far by any generator, for any action by the group. However, the methods in [BF95] and [Mor86] do not give an explicit bound. The technical adjective *ineffective* describes such non-constructive results. In contrast, if a proof is constructive or yields explicit quantities, then it is termed *effective*. The main result of

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this paper gives an estimate for this uniform bound in the case of Coxeter groups, in terms of the displacement function.

Given a finite presentation of a group G with generators $\{g_i\}$, and a representation $\rho : G \rightarrow \text{Isom}(\mathbb{H}^n)$, we define the displacement function of the action corresponding to ρ as the “mini-max” function $\inf_{x \in \mathbb{H}^n} \{\max_i d(g_i x, x)\}$.

Effective Compactness Theorem for Coxeter groups. *Let G be a Coxeter group given by a standard presentation with k generators, and suppose that G admits an isometric discrete and faithful action on \mathbb{H}^n . There exists a function $C_n(k) \in O(k^4)$ so that either G has a virtually solvable special nontrivial splitting or the displacement function is bounded above by $C_n(k)$ for every discrete and faithful action of G on \mathbb{H}^n . (We state the function explicitly in Section 7.)*

This result is related to work by Delzant [Del95] and Barnard [Bar07]. Delzant [Del95] proved an effective compactness theorem for faithful representations of groups to Gromov-hyperbolic groups. Barnard [Bar07] proved an effective compactness theorem for surface groups acting on an arbitrary complete geodesic δ -hyperbolic space, which generalizes the Mumford Compactness Theorem to δ -hyperbolic spaces. Both Delzant and Barnard’s results rely on the assumption that the injectivity radius of the group action is bounded from below. (The result in this paper does not use such an assumption.)

Summary. We begin by recalling the definitions and properties related to Coxeter groups that we use. Section 1 reviews special subgroups of Coxeter groups and defines special splittings following Mihalik and Tschantz’s visual decompositions [MT07].

We then discuss the hyperbolic geometry lemmas needed for the result. In Section 2, we calculate an estimate $\Lambda(\epsilon, R)$ for the length of a geodesic segment in \mathbb{H}^n that guarantees that the midpoint of the segment is moved at most ϵ by an involution, if the translation distance for the endpoints is bounded above by a constant R . In Section 3, we show that the quasi-convex hull of a finite set X in \mathbb{H}^n is quasi-isometric to the Gromov approximating tree for X , which is an abstract tree. In Section 4, we construct a projection of the tree to a collection of geodesic segments in \mathbb{H}^n spanning the set X , called the “shadow” of the Gromov approximating tree. To show that the shadow is quasi-isometric to a Gromov approximating tree of X , we use the quasi-isometry from Section 3.

To relate special splittings to the geometry of the action, we describe a combinatorial framework for assigning labels to the vertices of a tree. Section 5 introduces the system by which we label vertices. In Section 6, we use the fixed points of a Coxeter group action to generate a Gromov approximating tree. We apply the labelling system to the Gromov approximating tree to produce splittings of the Coxeter group. Each edge of the tree yields a special splitting.

In Section 7, we combine the above to prove the main result. We show that when an edge of the Gromov approximating tree is sufficiently long, then the splitting produced by an edge is nontrivial and small. Given an action ρ , we find a lower bound on the displacement function of ρ that ensures that the associated Gromov tree contains such a sufficiently long edge. To do so, we apply the estimate $\Lambda(\epsilon, R)$ obtained in Section 2 to geodesics contained in the shadow of the approximating tree, setting ϵ to the Margulis constant for \mathbb{H}^n .

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1. COXETER GROUPS

1.1. Notation. We begin by laying out conventions that are used in this paper:

W	Coxeter group
S	generating set for W in a standard presentation
(W, S)	Coxeter system
$\Gamma(W, S)$	Coxeter diagram for system (W, S)

The terms above are defined in Section 1.2.

1.2. Overview of Coxeter groups. We briefly recall definitions and properties of Coxeter groups that are used in the remainder of the paper. The most relevant Coxeter group properties for this paper are: (1) the generators in the standard presentations have finite order (see Definition 1.1), and (2) each relator corresponds to a finite subgroup of W (see Remark 1.2). The length of a relator is inconsequential.

Definition 1.1. A group W is a Coxeter group if it admits a presentation of the form

$$\langle s_1, \dots, s_g \mid (s_i s_j)^{m_{ij}} \rangle.$$

where $m_{ij} \leq \infty$ and

- $m_{ij} = 1$ if and only if $i = j$
- $m_{ij} \geq 2$ when $i > j$.
- $m_{ij} = \infty$ if and only if the element $s_i s_j$ has infinite order.

We call this presentation a standard presentation.

Denote the set of generators $S = \{s_1, \dots, s_g\}$. We call the pair (W, S) a Coxeter system and S a fundamental set of generators. We say that the rank of a Coxeter system is the cardinality of S .

Remark 1.2. If $(s_i s_j)^{m_{ij}}$ is a relator in a presentation of W , then $\langle s_i, s_j \rangle$ is a finite dihedral group.

Later in the paper, we may abbreviate Coxeter system as system.

A Coxeter diagram is a graph that encodes the information given by a standard presentation of a Coxeter group. We denote the graph as $\Gamma(W, S)$. Its vertices correspond to the generators bijectively; each vertex is labelled with its corresponding generator. An edge exists between two vertices s_i and s_j if and only if m_{ij} is finite and $i \neq j$. This edge is labelled with the number m_{ij} . Every Coxeter group determines a Coxeter diagram, a graph whose edges are labelled by positive integers, and any such graph determines a Coxeter system.

Figure 1 shows the diagram corresponding to the reflection group about a hyperbolic quadrilateral with angles $\pi/2, \pi/3, \pi/4, \pi/4$. A standard Coxeter presentation for this group is

$$\langle s_1, s_2, s_3, s_4 \mid s_i^2, (s_1 s_2)^2, (s_2 s_3)^4, (s_3 s_4)^3, (s_4 s_1)^4 \rangle.$$

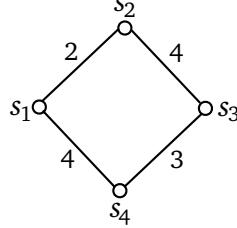


FIGURE 1. A sample Coxeter diagram.

Note that if a Coxeter diagram $\Gamma(W, S)$ is disconnected, then the Coxeter group can be expressed as a free product of the groups given by the components. For the remainder of the paper, we work only with Coxeter groups whose diagrams are connected. This assumption will be used in Section 6.2.

By abuse of notation, we may use s_i to denote the vertex of the Coxeter diagram that is labelled by the generator s_i as well as to denote the generator.

Note that the Coxeter diagram convention differs from the *Coxeter graph*, where edges are drawn if and only if $2 < m_{ij} \leq \infty$. The Coxeter group associated to a disconnected Coxeter graph can be decomposed into a direct product.

Coxeter diagrams are more common in geometric group theory, while Coxeter graphs are more common in Lie theory and combinatorics.

1.3. Special splittings.

Definition 1.3. Given a Coxeter system (W, S) , a subgroup W' of W is special if it is generated by a subset of S . The notation $(W', S') \subset (W, S)$ indicates that the subgroup W' is generated by $S' \subset S$.

As a set of vertices, a subset S' spans a unique maximal subdiagram $\Gamma' = \Gamma(W')$ of the Coxeter diagram $\Gamma(W)$ of (W, S) . We say that the subdiagram Γ' is special and call the associated subgroup $W(S') = \langle S' \rangle$.

A Coxeter system (W'', S'') is isomorphic to a special subgroup of (W, S) if there is an injection $j : W'' \hookrightarrow W$ carrying S'' to a subset of S . We call j a special injection.

Recall that when a group can be expressed as the amalgamated product $A *_C B$ or HNN-extension $A *_C$, we say that the group splits over C , and we refer to the amalgamated product or HNN-extension as a splitting. In this paper, we will not need to consider HNN-extensions. We let the injections defining an amalgamated product be denoted $i_A : C \hookrightarrow A$ and $i_B : C \hookrightarrow B$. A presentation of an amalgamated product is given by

$$A *_C B = \left\langle S(A) \cup S(B) \mid R(A) \cup R(B) \cup \bigcup_{s \in S(C)} \{i_A(s) = i_B(s)\} \right\rangle,$$

where the group A is given by the presentation $\langle S(A) \mid R(A) \rangle$, the group B is given by the presentation $\langle S(B) \mid R(B) \rangle$, and amalgamation subgroup C is generated by $S(C)$.

Definition 1.4. A splitting $W = A *_C B$ is a special splitting of a Coxeter system (W, S) if the following conditions hold:

- A, B , and C are special subgroups of W .

- The Coxeter diagram $\Gamma(C)$ is a subdiagram of $\Gamma(A)$ and $\Gamma(B)$.
- Let $j_A : A \hookrightarrow W$, $j_B : B \hookrightarrow W$, $j_C : C \hookrightarrow W$ be special injections for A , B , C into the Coxeter system (W, S) . Let Γ_A , Γ_B , and Γ_C be the subgraphs of $\Gamma(W)$ induced by the images of the special injection maps. Then the amalgamation maps $i_A : C \hookrightarrow A$ and $i_B : C \hookrightarrow B$ are induced by the inclusions of the subgraph Γ_C into Γ_A and Γ_B .

The conditions in Definition 1.4 are Mihalik and Tschantz's *visual axioms* for the case of splittings, or graphs of groups with one edge. The visual axioms for graphs of groups decompositions of Coxeter groups were introduced by Mihalik and Tschantz in [MT07], where the authors used special decompositions to show accessibility with respect to 2-ended splittings and to classify maximal FA-subgroups of finitely generated Coxeter groups.

Definition 1.5. A splitting is trivial if one of the amalgamation maps i_A or i_B is an isomorphism.

When the amalgamation groups A and B are Coxeter groups, and C is a special subgroup of A and B , then the group $A *_C B$ is a Coxeter group as well. Its diagram can be obtained by "visually amalgamating" the Coxeter diagrams for A and B (as graphs), in the following manner:

Definition 1.6. Suppose that $\alpha_A : \Gamma_C \hookrightarrow \Gamma_A$, $\alpha_B : \Gamma_C \hookrightarrow \Gamma_B$ are injections of the labelled simplicial graph Γ_C carrying edges to edges, vertices to vertices, forgetting vertex labels, and such that labels on edges are preserved. Then the labelled graph

$$\Gamma = \Gamma_A \cup \Gamma_B / \sim,$$

where $x \sim y$ when $x = \alpha_A \circ \alpha_B^{-1}(y)$ is called the visual amalgamation of the diagrams Γ_A and Γ_B over Γ_C . The edges of Γ inherit labels from Γ_A and Γ_B , and the vertices of Γ are unlabelled. We write $\Gamma = \Gamma_A \cup_{\Gamma_C} \Gamma_B$.

Given Coxeter systems (W_A, S_A) , (W_B, S_B) , (W_C, S_C) , let Γ_A , Γ_B , Γ_C be their associated Coxeter diagrams. Suppose that there are special injections from W_C to W_A and W_B given by $j_A : (W_C, S_C) \hookrightarrow (W_A, S_A)$ and $j_B : (W_C, S_C) \hookrightarrow (W_B, S_B)$. Let $L_A : S_A \rightarrow \text{vert}(\Gamma_A)$ be the bijection sending $s \in S_A$ to the vertex in Γ_A labelled s , and similarly for B . Let L_C be the bijection between S_C and vertices of Γ_C . Let α_A and α_B be defined as in Definition 1.6. Then the following diagram commutes:

$$\begin{array}{ccccc}
\text{vert}(\Gamma_A) & & \text{II} & & \text{vert}(\Gamma_B) \\
\uparrow & \searrow \alpha_A & & \swarrow \alpha_B & \uparrow \\
& \text{vert}(\Gamma_C) & & & \\
\downarrow L_A & & \downarrow L_C & & \downarrow L_B \\
S_A & \xrightarrow{j_A} & S_C & \xrightarrow{j_B} & S_B
\end{array}$$

Proposition 1.7. Let $W = W_A *_C W_B$, where the amalgamation is given by the maps j_A and j_B . Let (W', S') be the Coxeter system defined by the diagram Γ . Then $W \cong W'$.

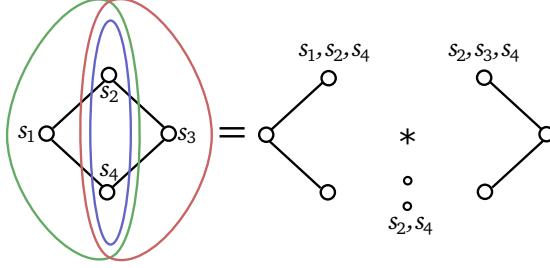


FIGURE 2. A sample splitting.

Proof. By inspection of the commutative diagram following Definition 1.6. \square

Definition 1.8. Let S' be a subset of generators for the system (W, S) . Suppose that there are subgroups $W_A \subset W$ and $W_B \subset W$, and maps $i_A : \langle S' \rangle \hookrightarrow W_A$ and $i_B : \langle S' \rangle \hookrightarrow W_B$ so that the amalgamated product $W_A *_{\langle S' \rangle} W_B \cong (W, S)$ determined by i_A and i_B is a special splitting. Then we say that the subset S' determines a special splitting of W .

Example 1.9. Suppose $(W, S) = \langle s_1, s_2, s_3, s_4 \mid \{s_i^2\}, \{(s_i s_{i+1})^{m_{i,i+1}}\} \rangle$ as in Figure 2. The subset $S' = \{s_1, s_3\}$ determines the splitting

$$\langle s_1, s_2, s_3 \rangle *_{\langle s_1, s_3 \rangle} \langle s_1, s_3, s_4 \rangle.$$

Remark 1.10. A subset does not always determine a unique special splitting, even when Γ is connected.

Proposition 1.11. Suppose that a Coxeter system (W, S) contains a special subgroup W' with diagram $\Gamma' \subset \Gamma$. Then the subgroup W' determines a special splitting of the system (W, S) if and only if the subdiagram Γ' separates the diagram Γ .

Proof. Suppose that Γ' separates Γ . Then there exist open nonempty disjoint subsets Γ'_A and Γ'_B of $\Gamma \setminus \Gamma'$ that cover $\Gamma \setminus \Gamma'$. Let Γ_A be the maximal subgraph of Γ spanned by $\text{vert}(\Gamma' \cup \Gamma'_A)$, and set $W_A = \langle \text{vert}(\Gamma_A) \rangle$, and similarly for B . By Proposition 1.7, the Coxeter system corresponding to the diagram $\Gamma = \Gamma_A \cup_{\Gamma'} \Gamma_B$ is $W_A *_{W'} W_B \cong W$.

Conversely, suppose that W' determines a special splitting. Let $W_A *_{W'} W_B$ be a special splitting of W . By Proposition 1.7, the Coxeter diagram for W is given by $\Gamma_A \cup_{\Gamma'} \Gamma_B$ with the amalgamation maps induced by the identity inclusion. Hence $\Gamma_A \cap \Gamma_B = \Gamma'$, so Γ' separates Γ . \square

We are ultimately interested in nontrivial special splittings. Recall from Definition 1.5 that trivial splitting occurs when at least one of the groups W_A or W_B equals W . This is the case if and only if $i_A : \langle S_C \rangle \hookrightarrow W_A$ or $i_B : \langle S_C \rangle \hookrightarrow W_B$ is an isomorphism, so one of S_A or S_B is the entire set S . Thus a trivial splitting occurs when one of the subdiagrams Γ_A or Γ_B is the entire diagram Γ .

2. BOUNDING THE MOVEMENT OF MIDPOINTS

The main result of this section is Proposition 2.3, which finds an estimate $\Lambda(\epsilon, R)$ for the length of a geodesic segment e in \mathbb{H}^n that guarantees the following: if an isometric involution of \mathbb{H}^n moves the endpoints of e at most distance R , then the midpoint of e is moved at most ϵ .

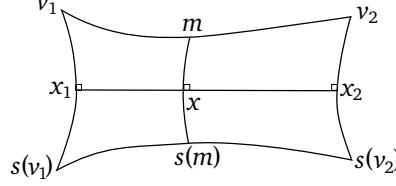


FIGURE 3. Diagram for Proposition 2.3.



FIGURE 4. Diagram for Lemma 2.6.

Recall that a group is *virtually* P if it contains a finite index subgroup with property P .

Definition 2.1. *A group is small if it is virtually solvable.*

Theorem 2.2 (Kazhdan-Margulis Theorem, [KM68]). *There exists a constant $\mu_n > 0$ (called the Margulis constant for \mathbb{H}^n) with the following property. Let $x \in \mathbb{H}^n$ and G be a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$ generated by $\{g_j\}$ such that*

$$d(x, g_j(x)) \leq \mu_n$$

for all j . Then the group G is small.

Proposition 2.3. (See Figure 4.) *Set $R \geq 0$. Let $e = [v_1 v_2]$ be a geodesic segment in \mathbb{H}^n and let s be an isometric involution of \mathbb{H}^n . Suppose that they satisfy*

$$d(v_1, s(v_1)), d(v_2, s(v_2)) \leq R.$$

Let m denote the midpoint of e . Define $\Lambda(\epsilon, R)$ as

$$(1) \quad \Lambda(\epsilon, R) = \frac{4}{\epsilon} + 2R.$$

Then for every $\epsilon > 0$, if $d(v_1, v_2) \geq \Lambda(\epsilon, R)$, then $d(m, s(m)) \leq \epsilon$.

The proof of Proposition 2.3 relies on the convexity of the distance function in hyperbolic space via the propositions that follow.

Here and in future sections, we use $[xy]_{\mathbb{H}} \subset \mathbb{H}^n$ to denote the geodesic path between two points x and y in \mathbb{H}^n . We denote the triangle $[xy]_{\mathbb{H}} \cup [yz]_{\mathbb{H}} \cup [zx]_{\mathbb{H}}$ by $[xyz]_{\mathbb{H}}$ and define a quadrilateral $[xyzw]_{\mathbb{H}}$ similarly. A quadrilateral $[xyzw]_{\mathbb{H}}$ may not necessarily be planar. Where there is no ambiguity, we may suppress subscripts.

Lemma 2.4 (Convexity of the hyperbolic distance function [BH99, II.2, Proposition 2.2]). *Let X be a geodesic metric space, and $c : [0, 1] \rightarrow X$ and $c' : [0, 1] \rightarrow X$ be two geodesics parametrized proportionally to arc length. Then for any $t, t' \in [0, 1]$, the maps c and c' satisfy the inequality*

$$d(c(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

The following is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Define c and c' as in Lemma 2.4, let I denote the interval $[0, 1]$, and let $C' = c'(I)$. Suppose $d(c(0), c'(0)), d(c(1), c'(1)) \leq r$. Then $d(c(t), C') \leq r$ for all $t \in I$.*

To obtain the estimate in Proposition 2.3, we use the function

$$h(x) = \sinh^{-1} \left(\frac{1}{\sinh(x)} \right),$$

where $x \in \mathbb{R}$ is strictly positive.

Lemma 2.6 ([Kap00, Lemma 3.5, pp. 34-35]). *(See Figure 4.) Let $[xyvw]$ be a quadrilateral in \mathbb{H}^n with angles $\angle wxy = \frac{\pi}{2}$, $\angle xyv = \frac{\pi}{2}$, $\angle xwv \geq \frac{\pi}{2}$. Then*

$$d(x, w) \leq h(d(x, y)).$$

Corollary 2.7. *Fix $\epsilon > 0, R \geq 0$. Let $[xyvw]$ be defined as in Lemma 2.6. Suppose that*

$$d(x, w), d(y, v) \leq \frac{R}{2} \text{ and } d(v, w) \geq h^{-1} \left(\frac{\epsilon}{2} \right) + R.$$

Then $d(x, w) \leq \frac{\epsilon}{2}$.

Proof of Lemma 2.7. We first note that h is decreasing. Let $g = h^{-1}(\frac{\epsilon}{2}) + R$. Since $d(v, w) \geq g$, we have

$$d(x, y) \geq g - (d(x, w) + d(y, v)) \geq g - R$$

and $h(d(x, y)) \leq h(g - R)$. We conclude that

$$d(x, w) \leq h(d(x, y)) \leq h(g - R) = h \left(h^{-1} \left(\frac{\epsilon}{2} \right) + R - R \right) = \frac{\epsilon}{2}.$$

□

Proof of Proposition 2.3. We show that a lower bound of $\Lambda(\epsilon, R)$ on the length of an edge e guarantees an upper bound on the movement of the midpoint m of e .

Let F_s denote the fixed-point set of the involution s . Let x_i denote the orthogonal projection of v_i to the fixed-point set F_s . Let $P : e \rightarrow [x_1 x_2]$ denote the orthogonal projection from e to the geodesic segment $[x_1 x_2]$, and m be the midpoint of e . Then $d(m, P(m)) \leq d(v_i, x_i) \leq \frac{R}{2}$ by Corollary 2.5.

Suppose that x_1 and x_2 are distinct points. Set $x = P(m)$. Either $\angle xmv_2 \geq \frac{\pi}{2}$ or $\angle xmv_1 \geq \frac{\pi}{2}$, since they are complementary angles. Without loss of generality, assume that $\angle xmv_2 \geq \frac{\pi}{2}$. Then $\angle xs(m)s(v_2) \geq \frac{\pi}{2}$, and the quadrilaterals $[xx_2v_2m], [xx_2s(v_2)s(m)]$ satisfy the conditions of Lemma 2.6.

Let $g = h^{-1}(\frac{\epsilon}{2}) + R$ as in Corollary 2.7. We assume that the length of $e = [v_1 v_2]$ is at least $2g$, so $d(m, v_2) = d(s(m), s(v_2)) \geq g$. Lemma 2.7 shows that $d(x, m) = d(x, s(m)) \leq \frac{\epsilon}{2}$. We conclude that

$$d(m, s(m)) \leq d(x, m) + d(x, s(m)) \leq \epsilon.$$

The case when x_1 and x_2 coincide follows by continuity.

We have shown that when the length of e is at least

$$2h^{-1} \left(\frac{\epsilon}{2} \right) + 2R,$$

the midpoint of e is moved no more than ϵ .

To complete the proof of the proposition, note that $h^{-1}(x) \leq \frac{1}{x}$. Hence it is sufficient to take the length of the edge e to be at least $\frac{4}{\epsilon} + 2R$ as desired. \square

3. THE QUASI-CONVEX HULL IS APPROXIMATELY A TREE

Here, we show that the “quasi-convex hull” of a finite subset of \mathbb{H}^n is quasi-isometric to the Gromov approximating tree for that subset.

Suppose X is a finite subset of \mathbb{H}^n . We define its quasi-convex hull $Q(X)$ as the union of geodesic segments between pairs of points of X :

$$Q(X) = \bigcup_{x,y \in X} [xy] \subset \mathbb{H}^n.$$

We refer to the segments comprising $Q(X)$ as edges of $Q(X)$.

Definition 3.1. Recall that two spaces X and Y are (L, A) -quasi-isometric if, for a given $L > 0, A \geq 0$, there is map $f : X \rightarrow Y$ such that the following are true:

(1) The map f satisfies

$$-A + \frac{1}{L}d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq Ld_X(x_1, x_2) + A$$

for all $x_1, x_2 \in X$.

(2) There is a map $\bar{f} : Y \rightarrow X$ such that

$$-A + \frac{1}{L}d_Y(y_1, y_2) \leq d_X(\bar{f}(y_1), \bar{f}(y_2)) \leq Ld_Y(y_1, y_2) + A$$

for all $y_1, y_2 \in Y$.

(3) The maps \bar{f} and f satisfy $d_X(x, \bar{f}f(x)) \leq A$ and $d_Y(y, f\bar{f}(y)) \leq A$.

If there is such an f , we call it an (L, A) -quasi-isometry. We say a map $f : X \rightarrow Y$ is a quasi-isometric embedding if it satisfies Property 1, but there is not necessarily a map $\bar{f} : Y \rightarrow X$ that satisfies Properties 2 and 3.

The main result of this section is:

Proposition 3.2. For any finite set $X \subset \mathbb{H}^n$, there is a finite metric tree T which is $(1, A)$ -quasi-isometric to $Q(X)$. We may take $A = (20c + 12) \ln 3$ for any $c > 0$ such that $|X| \leq 2^c + 2$. Thus A depends only on the cardinality of X . (We build the quasi-isometry $P : Q(X) \rightarrow T$ in Lemma 3.11.)

The proof of Proposition 3.2 uses the quasi-isometry of a hyperbolic triangle and its comparison tripod (Definition 3.3). We construct the quasi-isometry by first considering the union of maps from individual triangles in $Q(X)$ to tripods in T . The union forms a relation, and we refine the relation into the map P . We take T to be the Gromov approximating tree (see Definition 3.6).

Definition 3.3. Let $x_1, x_2, x_3 \in \mathbb{H}^n$, and let Δ denote the triangle $[x_1 x_2 x_3]$. The unique tripod τ_Δ with endpoints a_1, a_2, a_3 such that $d_{\mathbb{H}}(x_i, x_j) = d_T(a_i, a_j)$ for all i, j called the comparison tripod of Δ .

Let χ_Δ be the unique map sending $x_i \in X$ to $a_i \in \tau_\Delta$ for each i and which restricts to an isometry along each edge $[x_i x_j] \in X$.

Definition 3.4. Fix $\delta \geq 0$. We say that Δ is δ -thin if the preimage $y, z \in \chi_\Delta^{-1}(x)$ satisfies $d_X(y, z) \leq \delta$ for all $x \in X$.

Proposition 3.5 ([CDP90, 4.3]). *All triangles in real hyperbolic n -space \mathbb{H}^n are delta-hyperbolic for $\delta = \ln 3$.*

Hence we say that \mathbb{H}^n is δ -hyperbolic for $\delta = \ln 3$.

Definition 3.6. *Let X be a finite subset of \mathbb{H}^n with cardinality $|X| \leq 2^c + 2$ for $c > 0$. We say that a pair $(T, p : X \rightarrow T)$ is a Gromov approximating tree if T is a finite metric tree and $p : X \rightarrow T$ has the following properties:*

- (1) *The map $p : X \rightarrow T$ sends points in X to vertices of T , and*

$$\partial T \subset p(X),$$

where ∂T denotes the leaves of T (the valence-one vertices).

- (2) *The distance between pairs of points is quasi-preserved and p does not increase distance:*

$$d_{\mathbb{H}}(x_1, x_2) - 2c\delta \leq d_T(p(x_1), p(x_2))$$

for all $x_1, x_2 \in X$.

We sometimes abbreviate *Gromov approximating tree* to *approximating tree*.

Theorem 3.7 ([GD90, Section 2.1]). *Let X be a finite subset of \mathbb{H}^n and cardinality $|X| \leq 2^c + 2$ for $c > 0$. Then there exists a pair (T, p) with the properties described in Definition 3.6.*

Remark 3.8. *Gromov's construction applies to finite δ -hyperbolic spaces and finite sets of rays in δ -hyperbolic spaces.*

3.1. Triangles in $Q(X)$. Let X be a finite subset of \mathbb{H}^n with cardinality $|X| \leq 2^c + 2$ for $c > 0$, and let (T, p) be a Gromov approximating tree for X . Such a tree always exists by Theorem 3.7. We show below that triangles in $Q(X)$ are uniformly quasi-isometric to triangles in T by extending $p : X \rightarrow T$ to triangles in $Q(X)$.

Let $[xy]_T$ denote the geodesic segment between two points x, y in a tree T , and Let $[xyz]_T$ denote the tripod in T with leaves x, y, z . We suppress subscripts where there is no ambiguity.

Given $x_1, x_2, x_3 \in X$, let Δ be the triangle $[x_1x_2x_3] \subset \mathbb{H}^n$. Let

$$T_\Delta = [p(x_1)p(x_2)p(x_3)] \subset T$$

be the tripod in T with leaves $p(x_1), p(x_2), p(x_3)$. Let

$$o = [p(x_1)p(x_2)] \cap [p(x_2)p(x_3)] \cap [p(x_3)p(x_1)]$$

be the branch point of T_Δ . Let $\tau_\Delta = [a_1a_2a_3]$ be the comparison tripod (Definition 3.3) of Δ , and let $o_\tau = [a_1a_2] \cap [a_2a_3] \cap [a_3a_1]$ be the branch point of τ_Δ . Defining $\chi_\Delta : \Delta \rightarrow \tau_\Delta$ as in Definition 3.3, we call the points in $\chi_\Delta^{-1}(o_\tau)$ the *internal points*. Note that there is one internal point for each side $[x_i x_j]$. We label them o_{12}, o_{23}, o_{13} , where o_{ij} is contained in the side $[x_i x_j]$.

We extend $p|_{\{x_1, x_2, x_3\}}$ to the unique map $p_\Delta : \Delta \rightarrow T_\Delta$ that

- (1) sends x_i to $p(x_i)$ for each i
- (2) sends the point o_{ij} to o for each i, j
- (3) maps the segment $[x_i o_{ij}] \subset \Delta$ to the segment $[p(x_i)o] \subset T_\Delta$ as a dilation:

$$d_T(p(x_i), p_\Delta(x)) = \frac{d_T(p(x_i), o)}{d_{\mathbb{H}}(x_i, o_{ij})} \cdot d(x_i, x).$$

Proposition 3.9. *The map $p_\Delta : \Delta \rightarrow T_\Delta$ is a $(1, 4c\delta + 2\delta)$ -quasi-isometry, where Δ is given the subspace metric from $\Delta \subset \mathbb{H}^n$.*

Proof. The map p_Δ restricts to a $(1, 4c\delta)$ -quasi-isometric embedding along an edge $[x_i x_j]$ in Δ . To show this, we express the distance $d(p(x_i), o)$ as

$$d_T(p(x_i), o) = \frac{1}{2}(d_T(p(x_i), p(x_{i+1})) + d_T(p(x_i), p(x_{i-1})) - d_T(p(x_{i+1}), p(x_{i-1}))),$$

and combine this with the equations

$$d_{\mathbb{H}}(x_i, o_{ij}) = \frac{1}{2}(d_{\mathbb{H}}(x_i, x_{i+1}) + d_{\mathbb{H}}(x_i, x_{i-1}) - d_{\mathbb{H}}(x_{i+1}, x_{i-1}))$$

and

$$d(x_i, x_j) - 2c\delta \leq d(p(x_i), p(x_j)) \leq d(x_i, x_j).$$

To complete the proof, we consider any $x, y \in \Delta$. In a δ -thin triangle, it is always possible to find $x' \in \Delta$ and $y \in \Delta$ so that x', y' lie along a common side and $d_{\mathbb{H}}(x, x') \leq \delta$ and $d_{\mathbb{H}}(y, y') \leq \delta$. Hence the map p_Δ is a $(1, 4c\delta + 2\delta)$ -quasi-isometric embedding. Since p_Δ is surjective, it is a quasi-isometry. \square

3.2. Constructing the quasi-isometry between $Q(X)$ and T . To construct the map needed for Proposition 3.2, we use the triangle maps constructed in Section 3.1 to build a relation $P' : Q(X) \rightarrow T$, and then refine the relation into the desired map.

We define the relation

$$P' : Q(X) \rightarrow T$$

as follows: given $x \in Q(X)$, the image of x is the set

$$\{p_\Delta(x) \in T \mid x \in \Delta \subset Q(X)\}$$

where p_Δ is defined as in Section 3.1. Note that for each $x \in X$, the point $p(x)$ is contained in the image set $P'(x)$.

Proposition 3.10. *The relation $P' : Q(X) \rightarrow T$ satisfies the following three properties:*

- (1) *There exists a constant A' that uniformly bounds the diameter of $P'(x)$ for all x .*
- (2) *There exists A so that for each pair of points $x_1, x_2 \in Q(X)$, we can find $y_1 \in P'(x_1)$ and $y_2 \in P'(x_2)$ so that*

$$d(x_1, x_2) - A \leq d(y_1, y_2) \leq d(x_1, x_2) + A.$$

We may take $A = 4c\delta + 4\delta$ and $A' = 8c\delta + 4\delta$, so A and A' depend only on the cardinality of X .

- (3) *The relation P' is surjective.*

If we can show Proposition 3.10, when we can use the following to complete the proof of Proposition 3.2.

Lemma 3.11. *Let e be an edge of $Q(X)$. Suppose a relation $P' : Q(X) \rightarrow T$ satisfies the conditions of Proposition 3.10. Then there is a quasi-isometry $P : Q(X) \rightarrow T$ so that $P(x) \in P'(x)$ for all x , the vertices of T are contained in the image of P , and P is continuous on e .*

Proof of Lemma 3.11. We construct a map P , then use the assumed conditions of Proposition 3.10 to show that it satisfies the desired quasi-isometric inequalities.

First note that by construction of p_Δ , when $x \in X$, we have $P'(x) = p(x)$ (see Section 3.1). So, to construct P from P' , we pick images for points along edges $[x_1 x_2] \subset Q(X)$, where $x_1, x_2 \in X$. Let the vertices of X be $\{x_1, \dots, x_k\}$. Since X is

a finite set, there are finitely many edges in $Q(X)$; we order the edges e_1, e_2, e_3, \dots , etc. Without loss of generality, suppose that $e = e_1$ and x_1 is a boundary vertex of e_1 . We may choose the edges so that $e_i = [x_1 x_i]$ for $2 \leq i \leq k$.

For each edge e_i in $Q(X)$, we pick a triangle Δ_i containing e_i . Then, for each point $x \in e_i \setminus \{e_1 \cup \dots \cup e_{i-1}\}$, we set $P(x) = p_{\Delta_i}(x)$. One can check that the image of the map $P : Q(X) \rightarrow T$ thus defined contains all vertices of T . By construction, P is continuous on e .

To show that P is a quasi-isometry, let $x_1, x_2 \in Q(X)$. Combining Property 1 and Property 2 of Proposition 3.10, we have

$$d(x_1, x_2) - (A + 2A') \leq d(P(x_1), P(x_2)) \leq d(x_1, x_2) + (A + 2A').$$

Hence P is a $(1, A + 2A')$ -quasi-isometric embedding. Property 3 of Proposition 3.10 allows us to construct a quasi-inverse map; to show that the quasi-inverse inequalities are satisfied, we use Properties 1 and 2 of 3.10. \square

Proof of Proposition 3.10. To show Property (1), let $x \in Q(X)$. Let $\Delta_1, \Delta_2 \subset Q(X)$ be two triangles containing x , and suppose that $[x_1 x_2] \subset \Delta_1, \Delta_2$ is the edge of $Q(X)$ containing x . By Proposition 3.9,

$$d_{\mathbb{H}}(x, x_1) - 4c\delta - 2\delta \leq d_T(p_{\Delta_j}(x), x_1) \leq d_{\mathbb{H}}(x, x_1) + 4c\delta + 2\delta.$$

Since $p_{\Delta_1}(x)$ and $p_{\Delta_2}(x)$ lie along the geodesic $[p(x_1)p(x_2)]$, it follows that

$$d(p_{\Delta_1}(x), p_{\Delta_2}(x)) \leq 8c\delta + 4\delta.$$

Hence we may take $A' = 8c\delta + 4\delta$.

To show Property (2), let $x_1, x_2 \in Q(X)$. If x_1 and x_2 lie along edges that share a boundary vertex $x \in X$, then the points x_1 and x_2 lie in a triangle $\Delta \subset Q(X)$. In this case, Property (2) follows from Proposition 3.9. If x_1 and x_2 lie on disjoint edges in $Q(X)$, then they lie on opposite sides of a quadrilateral in $Q(X)$. In this case, it is possible to find two points x'_1 and x'_2 in a common triangle so that $d_{\mathbb{H}}(x_i, x'_i) \leq \delta$. Let Δ' be a triangle containing x'_1 and x'_2 . Then

$$d(x_1, x_2) - 4c\delta - 4\delta \leq d(p_{\Delta'}(x_1), p_{\Delta'}(x_2)) \leq d(x_1, x_2) + 4c\delta + 4\delta.$$

To show Property (3), let $x_1, x_2 \in X$ and $\Delta \subset Q(X)$ be a triangle in $Q(X)$ containing $[x_1 x_2]$. Then $P'([x_1 x_2])$ is the geodesic segment $[p(x_1)p(x_2)]_T$ of T . Since $p(X)$ contains all the leaves of T , the image $P'(Q(X))$ covers all geodesic segments between leaves of T . Hence P' is surjective. \square

Proof of Proposition 3.2. Let $P : Q(X) \rightarrow T$ be the map constructed in Lemma 3.11. Then P is an extension of p . It follows from Proposition 3.10 and Lemma 3.11 that P is a $(1, 20c\delta + 12\delta)$ -quasi-isometry between $Q(X)$ and $P(Q(X))$. \square

What we have shown can be summarized as:

Proposition 3.12. *Let e be an edge in $Q(X)$. Then one can find a map $P_e : Q(X) \rightarrow T$ that is continuous on e , and with the property that given $x \in Q(X)$, there is a triangle $\Delta_x \subset Q(X)$ where $x \in \Delta_x$ and $P_e(x) = p_{\Delta_x}(x)$. The map P_e is an extension of p , all vertices of T are contained in the image of P_e , and P_e is a $(1, 20c\delta + 12\delta)$ -quasi-isometry between $Q(X)$ and T .*

4. THE SHADOW OF AN APPROXIMATING TREE IN \mathbb{H}^n

The purpose of this section is to define a projection of T into \mathbb{H}^n , called the “shadow” T_{sh} of T . The shadow is a collection of geodesic segments in \mathbb{H}^n , and contains X .

To set up the definition of the shadow, we define a subset $\bar{X} \subset Q(X)$ as follows: let $V(T)$ denote the vertices of T , and let $P : Q(X) \rightarrow T$ be a map satisfying the conditions of Proposition 3.12. Then, given $y \in V(T)$, we assign to y a point $x \in Q(X)$ so that $P(x) = y$, with the requirement that if $y \in P(X)$, then the chosen x is an element of X . It is always possible to arrange the assignment so that no two points in $V(T)$ are assigned to the same x . This follows by construction of the map P and the definition of the Gromov approximating tree. Denote the set of points chosen as \bar{X} . The assignment gives a bijection, which we denote as $q_V : V(T) \rightarrow \bar{X}$. We extend q_V to a map q , which will allow us to define the “shadow” of T .

Proposition 4.1. *Let $q : T \rightarrow \mathbb{H}^n$ be the extension of $q_V : V(T) \rightarrow \bar{X}$ which is the unique map sending the edge $[y_1y_2] \subset T$ to $[q_V(y_1)q_V(y_2)] \subset \mathbb{H}^n$ via dilation: given $y \in [y_1y_2]$, we have*

$$d_{\mathbb{H}}(q_V(y_1), q_V(y)) = \frac{d_{\mathbb{H}}(q_V(y_1), q_V(y_2))}{d_T(y_1, y_2)} \cdot d_T(y_1, y).$$

Along each edge in T , the map q restricts to a $(1, 20c\delta + 12\delta)$ -quasi-isometry.

Proof. If y_1, y_2 are vertices of T , then $P(q_V(y_i)) = y_i$, for $i = 1, 2$. Hence we may apply Proposition 3.12 to the map $P : Q(X) \rightarrow T$ restricted to the points y_1 and y_2 , yielding the desired quasi-isometric inequality. \square

Remark 4.2. *The map q is in fact a $(1, |X|(20c\delta + 12\delta))$ -quasi-isometry between T and $q(T)$.*

Definition 4.3. *We call a point $x = q(y) \in q(T)$ the shadow of $y \in T$.*

To define the “shadow” of the tree T , we first observe that the image $q(T)$ contains the set \bar{X} . Recall the map $p : X \rightarrow T$ (Definition 3.6). When p is not injective, then \bar{X} does not contain the original set X generating the Gromov approximating tree. However, if $x \in X \setminus \bar{X}$, there is a unique element z of \bar{X} such that $p(x) = p(z)$.

For ease of exposition in later sections, we define the shadow of T to be a connected union of geodesic segments in \mathbb{H}^n containing $q(T)$ and X .

Definition 4.4. *We define the shadow of T , denoted T_{sh} , as the union of the image $q(T)$ and segments $[xz] \subset \mathbb{H}^n$ chosen as follows: if $x \in X$ is not an element of \bar{X} , then the segment $[xz]$ is included in T_{sh} , where z is the unique element of \bar{X} such that $p(x) = p(z)$.*

If x_1 and x_2 are points in T_{sh} , we define $d_{\text{sh}}(x_1, x_2)$ to be the distance of a shortest path in T_{sh} from x_1 to x_2 .

The shadow T_{sh} is a collection of geodesic segments $[xy] \subset \mathbb{H}^n$ whose combinatorics mimic those of T .

We let $[xy]_{\text{sh}}$ denote the following union of segments in T_{sh} : let z_1, z_2 be the unique elements of \bar{X} such that $p(x) = p(z_1)$ and $p(y) = p(z_2)$. Then we define $[xy]_{\text{sh}}$ as the concatenation of the segments $[xz_1], q([p(x)p(y)]),$ and $[yz_2]$.

5. LABELLING SYSTEMS

In this section, we develop a purely combinatorial framework for working with special splittings, called a *labelling system*. Labelling systems are a collection of labels for vertices of a tree; in Section 6.2, we will use them to produce special splittings from edges of an approximating tree. Nontrivial splittings correspond to *useful edges*; trivial splittings correspond to *useless edges*. Whether an edge is useful or useless can be determined combinatorially.

The main result for this section is Proposition 5.9, which says that the union of useful edges is connected. The crux of the proof of Proposition 5.9 is Proposition 5.8, which is essentially the Topological Helly Theorem, applied to the context of labelling systems.

5.1. Labelling systems. Let T be a finite simplicial tree. Recall that a valence-one vertex is a *leaf* and the set of leaves is ∂T . Recall that $[ab] \subset T$ denotes the minimum length path between vertices a and b of T .

Definition 5.1. A labelling of T is a relation $\text{Lab} : \text{vert}(T) \rightarrow \{1, \dots, N\}$. In particular, a vertex may have zero or more than one labels.

Definition 5.2. Let $\text{Lab}(v)$ be the set of labels assigned to a vertex v . We say that a relation

$$\text{Lab} : \text{vert}(T) \rightarrow \{1, \dots, N\}$$

is a labelling system if it satisfies the following properties:

Property A (connectedness). Let a and b be vertices of T , and let $x \in \text{vert}(T)$ be a vertex contained in the path $[ab] \subset T$. Then $\text{Lab}(a) \cap \text{Lab}(b) \subset \text{Lab}(x)$.

Property B (surjectivity). The full set of indices $\{1, \dots, N\}$ is contained in

$$\bigcup_x \text{Lab}(x),$$

where the union is taken over all vertices of T .

The labelling system used in Section 6.2 is constructed from an existing labelling as follows.

Definition 5.3. Let (T, Lab) be a labelled tree, and let V denote the vertices of T . We define a labelling $\overline{\text{Lab}} : V = \text{vert}(T) \rightarrow \{1, \dots, N\}$ as follows. Suppose x is a vertex of T . Let $Z(x)$ be the set of minimum-length paths in T passing through x , so $Z(x) = \{[ab] \mid x \in [ab]\}$. We set

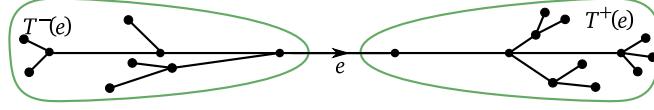
$$\overline{\text{Lab}}(x) = \bigcup_{[ab] \in Z(x)} (\text{Lab}(a) \cap \text{Lab}(b)),$$

so if x lies in the path $[ab]$ and a and b have a common label i , then $i \in \text{Lab}(x)$. We call $\overline{\text{Lab}}$ the canonical extension of Lab .

It follows that $\text{Lab}(x) \subset \overline{\text{Lab}}(x)$ for all vertices x in T .

Using standard techniques for working with paths in trees, one may verify the following two lemmas.

Lemma 5.4. Let T be a tree, Lab be a labelling of T , and $\overline{\text{Lab}}$ be the canonical extension of T . Suppose v is a vertex in $[ab]$ and $i \in \overline{\text{Lab}}(a) \cap \overline{\text{Lab}}(b)$. Then there exist a', b' such that $v \in [a'b']$ and $i \in \text{Lab}(a') \cap \text{Lab}(b')$.

FIGURE 5. The subtrees $T^+(e)$ and $T^-(e)$.

Lemma 5.5. *Let Lab be a labelling on T , and let $\overline{\text{Lab}}$ is the canonical extension of Lab . Then $\overline{\text{Lab}}$ is connected (Property A from Definition 5.2).*

It is an immediate consequence of Lemma 5.5 that:

Lemma 5.6. *If Lab is surjective (Property B from Definition 5.2), then $\overline{\text{Lab}}$ is a labelling system.*

We use the Lemmas 5.5-5.6 in Section 6.3, when we relate the geometry of approximating trees to the combinatorics of labellings.

5.2. Useless and useful edges. Suppose that (T, Lab) is a finite simplicial labelled tree, and that Lab is a labelling system (Definition 5.2).

The removal of any open edge $e \subset T$ separates the tree into two closed connected components. For the sake of bookkeeping, let us orient the edge. We call $T^+(e)$ the component toward which e is oriented and we call the remaining component $T^-(e)$, as illustrated in Figure 5.

Definition 5.7. *We say that an edge is useless if*

$$\bigcup_{v \in T^+} \text{Lab}(v) \quad \text{or} \quad \bigcup_{v \in T^-} \text{Lab}(v)$$

contains the full index set. An edge is useful if it is not useless.

Proposition 5.8. *Every edge of T is useless if and only if there exists a “full vertex”, i.e., a vertex z such that $\text{Lab}(z) = \{1, \dots, N\}$.*

Proof. Suppose that $z \in \text{vert}(T)$ is full. Let e be an edge of T . Then z is contained in either $T^+(e)$ or $T^-(e)$, so e is useless. Hence all edges are useless.

The other direction follows from the Topological Helly Theorem in [Deb70, Lemma A_m]. When working with a finite collection $\{T_i\}$ of contractible sets in a contractible space T , the Topological Helly Theorem states that if the space T has covering dimension 1 and the pairwise intersection $T_i \cap T_j$ is nonempty and connected for all $i \neq j$, then the intersection $\bigcap T_i$ is nonempty.

In our case, let T_i be the subtree of the tree T spanning all vertices labelled by i . By surjectivity of Lab (Definition 5.2, Property B), each T_i is nonempty. By construction, each T_i is contractible.

To show that T_i and T_j intersect, suppose by contradiction that they do not. Then there exists an edge e that separates T_i from T_j , i.e., an edge e such that $T_i \subset T^+(e)$ and $T_j \subset T^-(e)$. This contradicts the assumption that all edges are useless.

We have shown that $\bigcap T_i$ is nonempty. Because each T_i is a finite simplicial tree, and there are only a finite number of T_i , their intersection contains at least one vertex. Hence there exists a vertex labelled by all i in $\{1, \dots, N\}$. \square

Proposition 5.9. *The union of useful edges of T forms a subtree.*

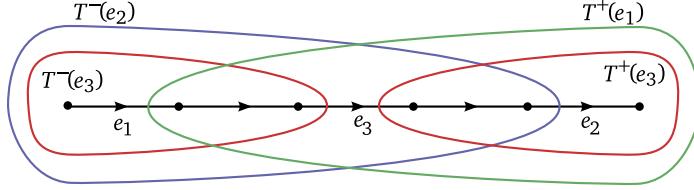


FIGURE 6. Useless edges cannot separate useful edges.

Proof. Let e_1 and e_2 be useful edges. Let e_3 be an edge contained in the unique geodesic path between e_1 and e_2 and orient the path so it flows from e_1 to e_2 (see Figure 6). We show that e_3 is useful.

Since e_1 and e_2 are useful edges, neither $T^+(e_1)$ nor $T^-(e_2)$ contain all labels. By way of contradiction, suppose that e_3 is a useless. Then the vertices of either $T^+(e_3)$ or $T^-(e_3)$ contain all the labels. However, because e_3 lies between e_1 and e_2 , we have $T^+(e_3) \subset T^+(e_1)$ and $T^-(e_3) \subset T^-(e_2)$. This means that either

$$\bigcup_{v \in T^+(e_3)} \text{Lab}(v) = S \subset \bigcup_{v \in T^+(e_1)} \subsetneq S$$

or

$$\bigcup_{v \in T^-(e_3)} \text{Lab}(v) = S \subset \bigcup_{v \in T^-(e_2)} \subsetneq S,$$

giving a contradiction. \square

Recall that our ultimate aim is to associate edges in a Gromov approximating tree to splittings. For the proof of the main result, we are interested in nontrivial splittings. As we show in Section 6.4, an edge produces a nontrivial splitting when it is *useful* in the sense of Definition 5.7. For this reason, we let T_{spl} denote the subtree formed by useful edges.

Proposition 5.10. *Let Lab_{spl} be the restriction of Lab to T_{spl} . For nonempty T_{spl} , the relation Lab_{spl} is surjective (Definition 5.2, Property B).*

Proof. We assume that T_{spl} is nonempty. To show surjectivity of Lab_{spl} , we construct a tree T' by collapsing the useful subtree to a point: $T' = T/T_{\text{spl}}$. Let $\rho_{\text{spl}} : T \rightarrow T'$ be the quotient map that induces the identification.

The image $v_{\text{spl}} = \rho_{\text{spl}}(T_{\text{spl}})$ is a vertex of T' . If w is a vertex of T' other than v_{spl} , it lifts to a unique vertex in $T \setminus T_{\text{spl}}$. Hence we define $\text{Lab}' : \text{vert}(T') \rightarrow \{1, \dots, N\}$ to send v_{spl} to $\bigcup_{v \in T_{\text{spl}}} \text{Lab}(v)$ and other vertices w to $\text{Lab}(\rho_{\text{spl}}^{-1}(w))$.

Since Lab is a labelling system, so is Lab' . Furthermore, every edge of T' is useless by construction. By Proposition 5.8, the tree T' has a full vertex x .

We claim that $x = v_{\text{spl}}$. By contradiction, suppose that is it not. Then

$$\text{Lab}(\rho_{\text{spl}}^{-1}(x)) = \{1, \dots, N\},$$

so the vertex $\rho_{\text{spl}}^{-1}(x)$ of T is full. By Proposition 5.8, the existence of a full vertex implies that all edges of T are useless. This is a contradiction, as T_{spl} is nonempty. Hence $x = v_{\text{spl}}$, and $\bigcup_{v \in T_{\text{spl}}} \text{Lab}(v) = \{1, \dots, N\}$. We conclude Lab_{spl} is a surjective relation. \square

6. BOUNDS ON THE DISPLACEMENT FUNCTION

6.1. The space of discrete and faithful representations and the displacement function. An isometric action $G \curvearrowright \mathbb{H}^n$ is equivalent to a representation $\rho : G \rightarrow \text{Isom}(\mathbb{H}^n)$; the *representation variety* of G -actions on \mathbb{H}^n is defined as

$$R(G, n) = \text{Hom}(G, \text{Isom}(\mathbb{H}^n)) = \{\rho : G \rightarrow \text{Isom}(\mathbb{H}^n)\}.$$

We define the adjoint action $ad \curvearrowright \text{Isom}(\mathbb{H}^n)$ of $\text{Isom}(\mathbb{H}^n)$ on itself via conjugation: $ad(h) \cdot h' = h^{-1}h'h$. The adjoint action induces an action $\text{Isom}(\mathbb{H}^n)$ on $R(G, n)$; for $\rho \in R(G, n)$ and $h \in \text{Isom}(\mathbb{H}^n)$, the representation $h \cdot \rho$ sends g to $h^{-1}\rho(g)h$. The space of conjugacy classes of representations in $R(G, n)$ is homeomorphic to the quotient

$$\overline{R}(G, n) = R(G, n) / \text{Isom}(\mathbb{H}^n),$$

where $\text{Isom}(\mathbb{H}^n)$ acts on $R(G, n)$ by the action induced by ad .

Unfortunately, the above space is in general non-Hausdorff [Kap00, Section 4.3, p. 57]. So one instead considers the Mumford quotient

$$X(G, n) = \text{Hom}(G, \text{Isom}(\mathbb{H}^n)) // \text{Isom}(\mathbb{H}^n),$$

which is an algebraic variety. The space $X(G, n)$ is called the *character variety*. For more information on this space, see [Mor86]. The series of work by Culler, Morgan, and Shalen [CS83][MS84][MS88a][MS88b][Mor86] examine the character variety.

We are interested in conjugacy classes of discrete and faithful actions on \mathbb{H}^n . Let

$$\text{Hom}_{\text{df}}(G, n) \subset \text{Hom}(G, \text{Isom}(\mathbb{H}^n))$$

denote the space of discrete and faithful representations. When G does not contain any infinite nilpotent normal subgroups (e.g., it is not small), then

$$\text{Hom}_{\text{df}}(G, n) / \text{Isom}(\mathbb{H}^n)$$

is Hausdorff, and in particular, it is a subspace of both $\overline{R}(G, n)$ and the character variety $X(G, n)$ (see [Kap00, Chapter 8, p. 157]).

Definition 6.1. *Given a group G and a dimension n , we define*

$$\mathcal{D}(G, n) = \text{Hom}_{\text{df}}(G, n) / \text{Isom}(\mathbb{H}^n),$$

and call this set the deformation space of G into $\text{Isom}(\mathbb{H}^n)$.

Definition 6.2. *Let G be a finitely-presented group generated by S . Let $\rho : G \rightarrow \text{Isom}(\mathbb{H}^n)$ be a representation, and let $B_\rho(x) = \max_{s \in S} d(x, s(x))$. The displacement function of a representation is defined as $B_\rho = \inf_{x \in \mathbb{H}^n} B_\rho(x)$. We denote the supremum of displacement functions of representations in a deformation space as*

$$B = \sup_{\rho \in \mathcal{D}(G, n)} B_\rho.$$

Given $[\rho], [\rho'] \in \mathcal{D}(G, n)$, we have $B_\rho = B_{\rho'}$ when $[\rho] = [\rho']$.

In [Bes88], Bestvina observed that the Compactness Theorem is equivalent to a uniform upper bound on the displacement function. As discussed in the introduction, the methods used to prove the Compactness Theorem do not give estimates for such a bound in general.

6.2. Application of combinatorial framework to special splittings. Let W be a Coxeter group with system (W, S) , and $\rho : W \curvearrowright \mathbb{H}^n$ be a discrete, faithful, and isometric action. We associate this data to an approximating tree T . Below, we define the subset X of \mathbb{H}^n from which T is constructed.

We assume that the Coxeter diagram $\Gamma(W, S)$ is connected; a disconnected Coxeter diagram corresponds to a splitting over the trivial group, which is small.

Let $S = \{s_1, \dots, s_k\}$. Consider the set \mathcal{S} of pairs $\{s_i, s_j\}$ which generate finite dihedral groups. Being finite, these dihedral groups have nonempty fixed-point sets in \mathbb{H}^n (see [BH99, Chapter II.6, Proposition 6.7]).

Fix a representation $\rho : W \rightarrow \text{Isom}(\mathbb{H}^n)$, and suppose it is discrete and faithful. Let X be a set of representative points from the fixed-point sets of pairs in \mathcal{S} ; hence $|X| \leq |\mathcal{S}| \leq \binom{k}{2}$.

Remark 6.3. *The space X can have arbitrarily large diameter, even in the case $k = 3$.*

Let (T, p) be a Gromov approximating tree for the set X , and recall the map $q : T_{\text{sh}} \rightarrow T$. Let $\text{stab}_S(x)$ denote the set of elements in S that fix x .

Definition 6.4. *Let the labelling $\text{Lab} : V(T) \rightarrow S$ send a vertex v to the union of labels $\bigcup_{x \in p^{-1}(v)} \text{stab}_S(x)$ when $q(v) \in X$ and to the empty set otherwise. We define a generator labelling denoted $\overline{\text{Lab}} : V(T) \rightarrow S$ as the canonical extension of Lab .*

Theorem 6.5. *The labelling $\overline{\text{Lab}} : V(T) \rightarrow S$ is a labelling system.*

Proof. We show that the map $\overline{\text{Lab}}$ is connected and surjective (Definition 5.2, Property A and Property B). The properties essentially hold by construction.

Property A follows from Lemma 5.5.

To show Property B, note that each $\rho(s_i)$ has a nonempty fixed-point set because it is an involution. The diagram for (W, S) is connected, so there exists an $s_{j \neq i}$ such that $\{s_i, s_j\}$ generate a finite dihedral group, which has nonempty fixed point set. For each s_i , there is a point $x_i \in X$ fixed by s_i . Thus there is a point in $V(T)$ labelled by s_i , namely, the point $p(x_i)$. Since $p(X) \subset V(T)$, it follows that the union $\bigcup_{v \in V(T)} \overline{\text{Lab}}(v)$ contains the full set $S = \{s_1, \dots, s_g\}$, so $\overline{\text{Lab}}$ is surjective. \square

6.3. Correspondence between the combinatorics of labelling systems and the geometry of actions. Let $\text{Lab} : V(T) \rightarrow S$ and $\overline{\text{Lab}} : V(T) \rightarrow S$ be the maps defined in Definition 6.4.

The combinatorics of the generator labelling system $\overline{\text{Lab}}$ correspond to the geometry of the action: if s_i labels a vertex $v \in T$, then we can bound the amount by which $\rho(s_i)$ displaces its shadow $q(v)$. To make this statement precise, we introduce the notion of an R -fixed point.

Definition 6.6. *Let $R \geq 0$. Fix an action $W \curvearrowright \mathbb{H}^n$. We say a point x is R -fixed by elements w_1, \dots, w_m of W if*

$$d(x, w_i(x)) \leq R \text{ for all } i \in \{1, \dots, m\}.$$

Proposition 6.7. *Let $s \in S$. Suppose u, w are vertices in T and x, y are vertices in T_{sh} so that $P(x) = u$, $P(y) = w$, and $s \in \text{Lab}(u) \cap \text{Lab}(w)$. If $z \in [xy]_{\text{sh}}$, then z is R -fixed by s , where $R = 2^8(|X|(20c\delta + 12\delta) + 4c\delta)$.*

As a consequence of Proposition 6.7, we obtain:

Proposition 6.8. *Suppose that v is a vertex of T and $s \in \overline{\text{Lab}}(v)$. Then $q(v)$ is R -fixed by s , where R is defined as in Proposition 6.7.*

Proof. If $s \in \overline{\text{Lab}}(v)$, then by Lemma 5.4, there are vertices $u, w \in T$ and $x, y \in T_{\text{sh}}$ such that $v \in [uw]$, $s \in \text{Lab}(u) \cap \text{Lab}(w)$, and $p(x) = u, p(y) = w$. Since x and y are fixed by s , it follows from Proposition 6.7 that the vertex $q(v)$ is R -fixed by s . \square

Proof of Proposition 6.7. We use the following lemma.

Lemma 6.9 ([Kap00, Lemma 3.43, pp. 48-49]). *Let $\gamma = [xy]_{\mathbb{H}}$ be a geodesic and $\widehat{\gamma}$ be an $(1, A)$ -quasi-geodesic path from x to y . Then $\widehat{\gamma}$ is contained in a regular r -neighbourhood $N_r(\gamma)$, where $r = 2^7 A$.*

Let $\gamma = [xy]_{\mathbb{H}}$ and $\widehat{\gamma} = [xy]_{\text{sh}}$. We claim that $\widehat{\gamma}$ is a $|X|(20c\delta + 12\delta) + 4c\delta$ quasi-geodesic. To show this, let $P_{\gamma} : Q(X) \rightarrow T$ be a map sending γ to $[uw]$ satisfying the conditions of Lemma 3.11. The map P_{γ} is distance decreasing on elements of X . Let z_1 and z_2 be the unique points of \overline{X} such that $p(x) = p(z_1)$ and $p(y) = p(z_2)$. Let $\widehat{\gamma}'$ be the segment of $\widehat{\gamma}$ contained in the image of q ; it is a sequence of edges of T_{sh} connecting z_1 to z_2 . Recall that when restricted to each edge, the map q is a $(1, 20c\delta + 12\delta)$ -quasi-isometry which is a homeomorphism. Thus, we can construct a $(1, |X|(20c\delta + 12\delta))$ -quasi-isometry $\alpha : \widehat{\gamma}' \rightarrow [uw]$ by defining α as q^{-1} along edges. Then $q \circ P_{\gamma}$ sends γ to $\widehat{\gamma}'$, so the length of $\widehat{\gamma}'$ is at most $d_{\mathbb{H}}(x, y) + |X|(20c\delta + 12\delta)$.

By construction of T_{sh} (Definition 4.4) and the definition of a Gromov approximating tree (Definition 3.6, Property 2), points in T_{sh} not contained in $q(T)$ are at most $2c\delta$ away from $q(T)$. Since the shortest path between x and y is at least as long as $d_{\mathbb{H}}(x, y)$, it follows that

$$d_{\mathbb{H}}(x, y) \leq d_{\text{sh}}(x, y) \leq d_{\mathbb{H}}(x, y) + |X|(20c\delta + 12\delta) + 4c\delta.$$

Set $A = |X|(20c\delta + 12\delta) + 4c\delta$. Let a be a point on $\widehat{\gamma}$. Let b be the nearest point on γ to a . The element s fixes x and y , so s fixes γ and sends $[ab]_{\mathbb{H}}$ to $[s(a)b]_{\mathbb{H}}$. The distance $d(a, b) = d(s(a), b)$ is bounded above by $r = 2^7 A$ as a consequence of Lemma 6.9. It follows that $d(a, s(a))$ is bounded above by $R = 2^8 A$.

We have shown that all points of $\widehat{\gamma}$ are R -fixed by s . \square

6.4. Special splittings produced by Gromov approximating trees. An edge e of the Gromov approximating tree T determines a special splitting in the following way.

Proposition 6.10. *Define sets of generators*

$$S^+(e) = \bigcup_{v \in T^+(e)} \{s \in \text{Lab}(v)\},$$

$$S^-(e) = \bigcup_{v \in T^-(e)} \{s \in \text{Lab}(v)\},$$

$$S^*(e) = S^+(e) \cap S^-(e).$$

*The special splitting $\langle S^+(e) \rangle *_{\langle S^*(e) \rangle} \langle S^-(e) \rangle$, where the amalgamation maps are induced by the inclusions $S^*(e) \hookrightarrow S^{\pm}(e)$, yields a group isomorphic to W . Furthermore, the splitting is trivial if and only if e is useless, i.e., either $S^+(e) = S$ or $S^-(e) = S$.*

Proof. Let Γ denote the Coxeter diagram for the system (W, S) . We let $\Gamma^* \subset \Gamma$ denote the subgraph spanned by $S^*(e)$; we define Γ^+ and Γ^- similarly.

By Proposition 1.11, it suffices to show that Γ^* separates Γ and that $S \subset S^+(e) \cup S^-(e)$.

We first show that Γ^* separates Γ . Let γ be a path from $\Gamma^+ \setminus \Gamma^*$ to $\Gamma^- \setminus \Gamma^*$. By way of contradiction, suppose that γ does not pass through Γ^* . Then γ contains vertices $s_+ \in S^+(e) \setminus S^*(e)$ and $s_- \in S^-(e) \setminus S^*(e)$ that are connected by exactly one edge in Γ . Hence the group $\langle s_+, s_- \rangle$ is a finite dihedral group, and $\{s_+, s_-\}$ is an element of S . So $\{s_+, s_-\}$ is contained in either $S^+(e)$ or $S^-(e)$. Without loss of generality, suppose that $\{s_+, s_-\} \subset S^+(e)$. Then $s_- \in S^+(e) \cap S^-(e) = S^*(e)$. This contradicts the assumption that $s_- \in S^-(e) \setminus S^*(e)$.

We have shown that $\langle S^+(e) \rangle *_{S^*(e)} \langle S^-(e) \rangle$ is a splitting. To complete the proof, it remains to show that $W \cong \langle S^+(e) \rangle *_{S^*(e)} \langle S^-(e) \rangle$. It suffices to check that $S^+(e) \cup S^-(e) = S$. This follows from the surjectivity of $\overline{\text{Lab}}$. Hence an edge e determines the desired splitting. \square

The splitting over e may be trivial. In Section 6.5, we find a lower bound on the displacement function of a representation to ensure the existence of a nontrivial splitting.

6.5. Lower bound on the displacement function to guarantee existence of non-trivial splittings.

Proposition 6.11. *If every edge of T determines a trivial splitting of W , then T_{sh} contains a point that is R -fixed by all generators in S . Hence if $B_\rho > R$, there exists a nontrivial splitting of W .*

Proof. Suppose that every edge of T determines a trivial splitting. Then every edge of T is useless. By Proposition 5.8, there exists a vertex $v \in T$ labelled by the full set $S = \{s_1, \dots, s_k\}$. By Proposition 6.8, the point $q(v) \in V(T_{\text{sh}})$ is R -fixed by S .

If a point in \mathbb{H}^n is R -fixed by S , then $B_\rho \leq R$. This follows from the definition of the displacement function and the definition of an R -fixed point. This means that when $B_\rho > R$, there exists an edge e which is useful and thus determines a nontrivial splitting of W . \square

By Proposition 5.9, the union of edges of T which determine non-trivial splittings of W is connected.

Definition 6.12. *Let T_{spl} denote the subtree consisting of the union of useful edges and call this subtree the useful subtree.*

Theorem 6.13 (Useful subtree theorem). *When $B_\rho > R$, the useful subtree T_{spl} is nonempty.*

Proof. By Proposition 6.11, there exists a useful edge e of T . Hence T_{spl} is nonempty. \square

7. SMALL DECOMPOSITIONS

In this section, we prove the main result of the paper.

Let μ_n be the Kazhdan-Margulis constant for \mathbb{H}^n (Theorem 2.2).

Effective Compactness Theorem for Coxeter groups. *Let (W, S) be a Coxeter system and suppose S has k elements. There exists a function $C_n(k)$ with the property that either W has a small special nontrivial splitting or the displacement function is bounded above by $C_n(k)$ for every discrete and faithful W -action on \mathbb{H}^n . We may take*

$$C_n(k) = R + 2 \binom{k}{2} \Lambda(\mu_n, R),$$

where $R = 2^8 \binom{k}{2} (20c\delta + 12\delta) + 4c\delta$, and $\Lambda(\mu_n, R) = 2 \binom{k}{2} \left(\frac{4}{\mu_n} + 2R \right)$ (as defined in Proposition 2.3).

As a function of k , the estimate $C_n(k)$ is of order k^4 .

Definition 7.1. *We say that the shadow length of an edge e in T is the length of the quasi-geodesic segment $q(e) \subset T_{\text{sh}} \subset \mathbb{H}^n$. We let $|q(e)|$ denote the shadow length of e . (See Proposition 4.1 for the definition of the map q .)*

We seek a condition on the shadow length of a useful edge e that guarantees the existence of a small nontrivial splitting of W (Definition 2.1).

Let $\Lambda_n = \Lambda(\mu_n, R)$, where R is defined as in Proposition 6.8. Given an action $\rho : W \curvearrowright \mathbb{H}^n$ and an edge e of T , we define $S^*(e)$ as in Section 6.4.

Lemma 7.2. *If $|q(e)| \geq \Lambda_n$, then $S^*(e)$ generates a small group and the special splitting associated with e is small.*

Proof. Let m be the midpoint of $q(e)$. We show that m is μ_n -fixed by $S^*(e)$. According to Proposition 6.8, if $s \in S^*(e)$, then the shadow of each vertex in e is R -fixed by s . Therefore, since $|q(e)| \geq \Lambda_n$, it follows from Proposition 2.3 that $d(m, s(m)) \leq \mu_n$ for all $s \in S^*(e)$. The representation $\rho(\langle S^*(e) \rangle) \subset \text{Isom}(\mathbb{H}^n)$ is discrete, so by the Kazhdan-Margulis Lemma (Theorem 2.2), the group $\rho(\langle S^*(e) \rangle)$ is small. Since ρ is a faithful representation, we conclude that $S^*(e)$ generates a small group. \square

Lemma 7.3. *There exists a function $C_n(k)$ with the following property. If the shadow length of every edge in T_{spl} is less than or equal to Λ_n , then each $x \in q(T_{\text{spl}})$ is $C_n(k)$ -fixed by S . Furthermore, we may take*

$$C_n(k) = R + 2 \binom{k}{2} \Lambda_n = 2^8 \left(\binom{k}{2} (20c\delta + 12\delta) + 4c\delta \right) + 2 \binom{k}{2} \Lambda_n.$$

Proof. If T_{spl} is empty, then the lemma holds trivially, so we work with nonempty T_{spl} .

Let x be a point in $q(T_{\text{spl}})$, and let s_i be an element of S . We analyse how far s_i displaces x .

By Proposition 5.10, there is a vertex v_i of T_{spl} with $s_i \in \overline{\text{Lab}}(v_i)$. Since $|q(e)| \leq \Lambda_n$ for each edge e of T_{spl} , the distance $d(x, q(v_i))$ is bounded above by $E\Lambda_n$, where E is maximum number of edges in a geodesic path in T_{spl} .

Set $R = 26c\delta + 12\delta$. The distance between $q(v_i)$ and $s_i(q(v_i))$ is bounded above by R , by Proposition 6.8. It follows that

$$d(x, s_i(x)) \leq 2d(x, q(v_i)) + R \leq R + 2E\Lambda_n.$$

Since the value of E is bounded above by $\binom{k}{2}$, we conclude that x is $C_n(k)$ -fixed by S , where $C_n(k) = R + 2 \binom{k}{2} \Lambda_n$. \square

It follows that if all edges of T_{spl} have length less than or equal to Λ_n , then $B_\rho \leq C_n(k)$.

Proof of the Effective Compactness Theorem for Coxeter Groups. Let $C_n(k)$ be the function defined in Lemma 7.3, and suppose that $B_\rho > C_n(k)$. Then \mathbb{H}^n contains no point $C_n(k)$ -fixed by S . The useful subtree theorem (Theorem 6.13) guarantees that T_{spl} is nonempty, as $B_\rho > C_n(k) > R$. Since T_{spl} is nonempty, Lemma 7.3 guarantees the existence of an edge e of T_{spl} whose shadow length is greater than Λ_n . By Proposition 6.10, the special splitting determined by $S^*(e)$ is nontrivial. By Lemma 7.2, this splitting is small.

Thus either W admits a small special nontrivial splitting or $B_\rho \leq C_n(k)$ for every $[\rho] \in \mathcal{D}(W, n)$. \square

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